

# MATURITY-INDEPENDENT RISK MEASURES

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ABSTRACT. The new notion of maturity-independent risk measures is introduced and contrasted with the existing risk measurement concepts. It is shown, by means of two examples, one set on a finite probability space and the other in a diffusion framework, that, surprisingly, some of the widely utilized risk measures cannot be used to build maturity-independent counterparts. We construct a large class of maturity-independent risk measures and give representative examples in both continuous- and discrete-time financial models.

## 1. INTRODUCTION

The abstract notion of a risk measure appeared first in [1] and [2]. The simple axioms set forth in [2] opened a venue for a rich field of research that shows no signs of fatigue. The main reason for such success is the fundamental need for quantification and measurement of risk. While the initial impetus came from the requirements of the financial and insurance industries, applications in a wide range of situations, together with a mathematical tractability and elegance of this theory, have promoted risk measurement to an independent field of interest and research. The early cornerstones include (but are not limited to) [11, 12 and 14]; see, also, [13] for more information.

The first notions of risk measures were all static, meaning that the time of measurement, as well as the time of resolution (maturity, expiry) of the risk were fixed. Soon afterwards, however, dynamic and conditional risk measures started to appear (see [3, 5, 6, 7, 10, 29 and 30], as well the book [13]).

Despite all the recent work in this wide area, there is still a number of theoretical, as well as practical, questions left unanswered. The one we focus on in the present paper deals with the problem one faces when the maturity (horizon, expiration date, etc.) associated

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with a particular risky position is not fixed. We take the view that the mechanism used to measure the risk content of a certain random variable should not depend on any a priori choice of the measurement horizon. This is, for example, the case in complete financial markets. Indeed, consider for simplicity the Samuelson (Black-Scholes) market model with zero interest rate and the procedure one would follow to price a contingent claim therein. The fundamental theorem of asset pricing tells us to simply compute the expectation of the discounted claim under the unique martingale measure. There is no explicit mention of the maturity date of the contingent claim in this algorithm, or, for that matter, any other prespecified horizon. Letting the claim's payoff stay unexercised for any amount of time after its expiry would not change its arbitrage-free price in any way.

It is exactly this property that, in our opinion, has not received sufficient attention in the literature. As one of the fundamental properties clearly exhibited under market completeness, it should be shared by any workable risk measurement and pricing procedure in arbitrary incomplete markets.

The incorporation of the maturity-independence property described above into the existing framework of risk measurement has been guided by the principle of minimal impact: we strove to keep new axioms as similar as possible to the existing ones for convex risk measures, and to implement only minimally needed changes. This led us to the realization that it is the domain of the risk measure that inadvertently dictates the use of a specific time horizon, and if we replace it by a more general domain, the maturity-independence would follow. Thus, our axioms are identical to the axioms of a replication-invariant convex risk measure, except for the choice of the domain which is *not* a subspace of a function space on  $\mathcal{F}_T$ , for some fixed time horizon  $T$ .

In addition to the novel axiom pertinent to maturity independence, a link to the notion of forward performance processes, recently proposed by M. Musiela and the first author (see [24, 25, 26, 27, 28]) is established. Indeed, focusing on the exponential case, it is shown that every forward performance process can be used to create an example of a maturity-independent risk measure. On one hand, this connection provides a useful and simple tool for (a non-trivial task of) constructing maturity-independent risk measures. On the other hand, we hope that it would give a firm decision-theoretic foundation to the theory of forward performances.

We start off by introducing the financial model, trading and no-arbitrage conditions, and recalling some well-known facts about risk measures. In section 3, we introduce the notion of a maturity-independent risk measure, argue for its feasibility and relevance, and give first examples. We also show, via two simple examples, that a naïve approach to the construction of maturity-independent risk measures can fail. Section 4 opens with the notion of a performance random field and goes on to describe the important class of forward performance processes. These objects are, in turn, used to produce a class of maturity-independent risk measures which we call *forward entropic risk measures*. Finally,

several special cases of these measures are mentioned and interpreted in section 5, and an independent example, set in a binomial-type incomplete financial model, is presented.

## 2. GENERALITIES ON THE FINANCIAL MARKET MODEL AND RISK MEASURES

### 2.1. Market Set-up, No-Arbitrage Conditions and Admissible Portfolios.

2.1.1. *The Model.* Let  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$  be a complete probability space, with the filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty)}$  generated by a  $d$ -dimensional Brownian motion  $(W_t)_{t \in [0, \infty)} = (W_t^1, \dots, W_t^d)_{t \in [0, \infty)}$  (and augmented by the  $\mathbb{P}$ -null sets). The evolution of the prices of risky assets is modeled by an Itô-process  $(S_t)_{t \in [0, \infty)} = (S_t^1, \dots, S_t^k)_{t \in [0, \infty)}$  of the form

$$dS_t^i = S_t^i \left( \mu_t^i dt + \sum_{j=1}^d \sigma_t^{ji} dW_t^j \right), \quad (2.1)$$

for  $t \geq 0$ ,  $i = 1, \dots, k$  and  $j = 1, \dots, d$ , where the processes  $(\mu_t^i)_{t \in [0, \infty)}$  and  $(\sigma_t^{ji})_{t \in [0, \infty)}$ , are  $\mathbb{F}$ -progressively measurable and uniformly bounded by a deterministic constant. The requirement of uniform boundedness can be replaced by a much less stringent one, but we choose not to pursue such a generalization for the sake of transparency.

We postulate the existence of a  $d$ -dimensional progressively-measurable process  $(\lambda_t)_{t \in [0, \infty)}$  such that

$$\sum_{j=1}^d \sigma_t^{ji} \lambda_t^j = \mu_t^i, \quad i = 1, \dots, k, \quad t \geq 0, \quad \text{a.s.} \quad (2.2)$$

While such  $\lambda$  does not need to be unique, we will always work with the representative chosen in a minimal way, namely,

$$\lambda_t^j = \sum_{i=1}^k (\sigma^+)^{ji}_t \mu_t^i, \quad t \geq 0, \quad \text{a.s.}, \quad (2.3)$$

where  $\sigma^+$  denotes the Moore-Penrose pseudo-inverse of the matrix  $\sigma$ . Effectively, our choice of the market price of risk (vector) process  $\lambda$  amounts to the solution of the matrix equation  $\sigma \lambda = \mu$  with the minimal Euclidean norm. We assume, in addition, that each component of the process  $\lambda^j$ , just like  $\sigma^{ji}$  and  $\mu^i$ , is uniformly bounded by a (deterministic) constant.

The existence of a liquid risk-free asset  $S^0$  is also postulated. As usual, we quote all asset-prices in the units of  $S^0$ . Operationally, this amounts to the simplifying assumption  $S_t^0 = 1$ ,  $t \geq 0$ , which will hold throughout.

2.1.2. *Portfolio processes.* A  $k$ -dimensional  $\mathbb{F}$ -progressive process  $\pi = (\pi_t^1, \dots, \pi_t^k)_{t \in [0, \infty)}$  is called a **portfolio (process)** if  $\sum_{i=1}^k \int_0^t (\pi_s^i)^2 ds < \infty$ , a.s., for all  $t \geq 0$ . A portfolio  $\pi$  is called **admissible** if there exists a constant  $a > 0$  (possibly depending on  $\pi$ , but not on

the state of the world) such that the **gains process**  $X^\pi = (X_t^\pi)_{t \in [0, \infty)}$ , defined as

$$X_t^\pi = \int_0^t \pi_s dS_s = \sum_{i=1}^k \int_0^t \pi_s^i dS_s^i, \quad t \geq 0,$$

is bounded from below by  $-a$ , i.e.,  $X_t^\pi \geq -a$ , for all  $t \geq 0$ , a.s. The set of all portfolio processes  $\pi$  whose gains processes  $X^\pi$  are admissible is denoted by  $\mathcal{A}$ . For technical reasons, which will be clear shortly, we introduce the set  $\mathcal{A}_{\text{bd}}$  of all portfolio processes  $\pi$  whose gains process  $X^\pi$  is uniformly bounded from above, as well as from below, i.e.,  $\mathcal{A}_{\text{bd}} = \mathcal{A} \cap (-\mathcal{A}) = \{\pi \in \mathcal{A} : -\pi \in \mathcal{A}\}$ .

**2.1.3. No Free Lunch with Vanishing Risk.** The natural assumption of no arbitrage is routinely replaced in the literature by the slightly stronger, but still economically feasible, assumption of *no free lunch with vanishing risk* (NFLVR). It was shown in the seminal paper [9] that, when postulated on finite time-intervals  $[0, t]$ ,  $t \in (0, \infty)$ , NFLVR is equivalent to the following statement: for each  $t \geq 0$ , there exists a probability measure  $\mathbb{Q}^{(t)}$ , defined on  $\mathcal{F}_t$ , with the following properties:

- (1)  $\mathbb{Q}^{(t)} \sim \mathbb{P}|_{\mathcal{F}_t}$ , where  $\mathbb{P}|_{\mathcal{F}_t}$  is the restriction of the probability measure  $\mathbb{P}$  to  $\mathcal{F}_t$ , and
- (2) the stock-price process  $S$  is a  $\mathbb{Q}^{(t)}$ -local martingale, when restricted to the interval  $[0, t]$ .

It is well-known that, under the assumptions we imposed on the coefficient processes  $\mu$  and  $\sigma$ , the condition of NFLVR, and, thus, the equivalent statement above, are automatically satisfied on finite intervals  $[0, t]$ ,  $t \in (0, \infty)$ . Therefore, for  $t \geq 0$ , the set of all measures  $\mathbb{Q}^{(t)}$  with the above properties is non-empty. We will denote this set by  $\mathcal{M}_t^e$ .

**2.1.4. Closed market models.** It is immediate that, for  $0 \leq s < t$ , we have the following relation

$$\mathcal{M}_s^e = \left\{ \mathbb{Q}^{(t)}|_{\mathcal{F}_s} : \mathbb{Q}^{(t)} \in \mathcal{M}_t^e \right\}.$$

The restriction map turns the family  $(\mathcal{M}_t^e)_{t \in [0, \infty)}$  into an inversely directed system. In general, such a system will *not* have an inverse limit, i.e., there will exist *no* set  $\mathcal{M}_\infty^e$  with the property that  $\mathcal{M}_t^e = \{\mathbb{Q}|_{\mathcal{F}_t} : \mathbb{Q} \in \mathcal{M}_\infty^e\}$ , for all  $t$ . In other words, even though the market may admit no arbitrage (NFLVR) on any finite interval  $[0, t]$ , arbitrage opportunities might arise if we allow the trading horizon to be arbitrarily long. In order to differentiate those cases, we introduce the notion of a closed market model:

**Definition 2.1.** A market model  $(S_t)_{t \in [0, \infty)}$  is said to be **closed** if there exists a set  $\mathcal{M}_\infty^e$  of probability measures  $\mathbb{Q} \sim \mathbb{P}$  such that, for every  $t \geq 0$ ,  $\mathbb{Q}^{(t)} \in \mathcal{M}_t^e$  if and only if  $\mathbb{Q}^{(t)} = \mathbb{Q}|_{\mathcal{F}_t}$  for some  $\mathbb{Q} \in \mathcal{M}_\infty^e$ .

*Remark 2.2.* Most market models used in practice are not closed. The simplest example is Samuelson's model, where the filtration is generated by a single Brownian motion

$(W_t)_{t \in [0, \infty)}$ , and the price of the risky asset satisfies  $dS_t = S_t(\mu dt + \sigma dW_t)$ , for some constants  $\mu \in \mathbb{R}$ ,  $\sigma > 0$ . For  $t \geq 0$ , the only element in  $\mathcal{M}_t^e$  corresponds to a Girsanov transformation. However, as  $t \rightarrow \infty$ , this transformation becomes “more and more singular” with respect to  $\mathbb{P}|_{\mathcal{F}_t}$ , and no  $\mathbb{Q}$  as in Definition 2.1 can be found (see [20], Remark on p. 193).

## 2.2. Convex risk measures.

**2.2.1. Axioms of convex risk measures.** One of the main reasons for the wide use and general acceptance of the theory of risk measures lies in its axiomatic nature. Only the most fundamental traits of an economic agent, such as risk aversion, are encoded parsimoniously into the axioms of risk measures. The resulting theory is nevertheless rich and relevant to the financial practice. The pioneering notion of a coherent risk measure (see [2]) has, soon after its conception, been replaced by a very similar, but more flexible, notion of a convex risk measure (introduced in [11, 14, 16 and 17]):

**Definition 2.3.** A functional  $\rho$  mapping  $\mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$  into  $\mathbb{R}$  is called a **convex risk measure** if, for all  $f, g \in \mathbb{L}^\infty$ , we have

- (1)  $\rho(f) \leq 0$  if  $f \geq 0$ , a.s.; (anti-positivity)
- (2)  $\rho(f - m) = \rho(f) + m$ ,  $m \in \mathbb{R}$ ; (cash-translativity)
- (3)  $\rho(\lambda f + (1 - \lambda)g) \leq \lambda \rho(f) + (1 - \lambda)\rho(g)$ ,  $\lambda \in [0, 1]$ . (convexity)

**2.2.2. Replication invariance.** The idea that two risky positions which differ only by a quantity replicable in the market at no cost, should have the same risk content has appeared very soon after the notion of a risk measure has been applied to the study of financial markets. In order to expand on this tenet, let us, temporarily, pick an arbitrary time  $T > 0$ , and suppose that we are dealing with a finite-horizon financial market  $(S_t)_{t \in [0, T]}$ , where all finite-horizon analogues of the assumptions and definitions above hold. In such a situation, the investors will trade in the market in order to reduce the overall risk of the terminal position, as measured by the risk measure  $\rho$  defined on  $\mathbb{L}^\infty(\mathcal{F}_T)$ . In other words, the combination of the financial market and the risk measure  $\rho$  will give rise to a new risk measure, denoted herein by  $\rho(\cdot; T)$ , given by

$$\rho(f; T) = \inf_{\pi \in \mathcal{A}_{\text{bd}}} \rho\left(f + \int_0^T \pi_s dS_s\right).$$

We will use the  $T$ -notation to stress the dependence of this risk measure on the specific maturity date. In addition to the Axioms (1)-(3) from Definition 2.3, the functional  $\rho(\cdot; T)$  satisfies the following property:

- (4)  $\rho(f; T) = \rho(f + \int_0^T \pi_s dS_s; T)$  for all  $f \in \mathbb{L}^\infty(\mathcal{F}_T)$ ,  $\pi \in \mathcal{A}_{\text{bd}}$ . (replication invariance)

**Definition 2.4.** A mapping  $\rho(\cdot; T) : \mathbb{L}^\infty(\Omega, \mathcal{F}_T, \mathbb{P}|_{\mathcal{F}_T}) \rightarrow \mathbb{R}$  is called a **replication-invariant convex risk measure** if it satisfies axioms (1)-(3) of Definition 2.3 and (4) above.

The notion of replication invariance was introduced in [11], and further developed and generalized in [15]. An accessible discussion of coherent and convex risk measures, as well as the notion of replication invariance, can be found in chapter 4 of [13].

*Remark 2.5.*

- (1) When the market model is complete, the restrictions imposed by adding the replication invariance axiom will necessarily force any replication-invariant risk measure to coincide with the replication price functional (the “Black-Scholes price”). It is only in the setting of incomplete markets that the interplay between risk measurement and trading in the market produces a non-trivial theory.
- (2) It may seem somewhat counterintuitive at the first glance that a replication-invariant risk measure should assign the same risk content to the constant  $S_0$  as to the random variable  $S_T$  (where  $(S_t)_{t \in [0, T]}$  is a price process of a traded risky asset). The resolution can be found in the fact that the risk contained in  $S_T$  is virtual since it can be hedged away completely in the financial market. Replication-invariant measures are, however, typically not law-invariant, i.e., there are random variables with the same  $\mathbb{P}$ -distribution as  $S_T$  whose risk content is possibly much larger.

The following examples of (maturity-specific) replication-invariant convex risk measures are very well known (see [13]). We use them as test cases for the notion of maturity independence introduced in Definition 3.1, below. One can easily show that all of them satisfy axioms (1)-(4).

**Example 2.6.**

- (1) **Super-hedging.** For  $f \in \mathbb{L}^\infty(\mathcal{F}_T)$ , let  $\hat{\rho}(f; T)$  be the super-hedging price of  $f$ , i.e.,

$$\hat{\rho}(f; T) = \inf \left\{ m \in \mathbb{R} : \exists \pi \in \mathcal{A}_{\text{bd}}, \int_0^T \pi_s dS_s \geq m + f, \text{ a.s.} \right\}.$$

The risk measure  $\hat{\rho}(\cdot; T)$  is extremal in the sense that, for each replication-invariant convex risk measure  $\rho(\cdot; T)$ , we have  $\hat{\rho}(f; T) \geq \rho(f; T)$ , for all  $f \in \mathbb{L}^\infty(\mathcal{F}_T)$ .

- (2) **Entropic risk measures.** For  $f \in \mathbb{L}^\infty(\mathcal{F}_T)$ , the entropic risk measure  $\rho(f; T)$ , with risk aversion coefficient  $\gamma > 0$ , is defined as the unique solution  $\rho \in \mathbb{R}$  to the indifference-pricing equation

$$\begin{aligned} \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E} \left[ -\exp \left( -\gamma \left( x + \rho + f + \int_0^T \pi_s dS_s \right) \right) \right] = \\ \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E} \left[ -\exp \left( -\gamma \left( x + \int_0^T \pi_s dS_s \right) \right) \right], \quad x \in \mathbb{R}. \end{aligned} \tag{2.4}$$

The value  $\rho(-f; T)$  at the negative  $-f$  of  $f$  is also known as the **exponential indifference price**  $\nu(f; T)$  of  $f$ . The measure  $\rho(\cdot; T)$  admits a simple dual representation

$$\rho(f; T) = \sup_{\mathbb{Q} \in \mathcal{M}_T^e} \left( \mathbb{E}^{\mathbb{Q}}[-f] - \frac{1}{\gamma} H(\mathbb{Q}|\mathbb{P}; T) \right), \quad (2.5)$$

where the **relative entropy**  $H(\mathbb{Q}|\mathbb{P}; T)$  of  $\mathbb{Q} \in \mathcal{M}_T^e$  **with respect to**  $\mathbb{P}$  is given by

$$H(\mathbb{Q}|\mathbb{P}; T) = \mathbb{E}^{\mathbb{Q}} \left[ \ln \left( \frac{d\mathbb{Q}}{d\mathbb{P}|_{\mathcal{F}_T}} \right) \right] \in [0, \infty].$$

- (3) **General replication-invariant risk measures.** Under appropriate topological regularity conditions replication-invariant convex risk measure  $\rho(\cdot; T) : \mathbb{L}^\infty(\mathcal{F}_T) \rightarrow \mathbb{R}$  admits the following dual representation

$$\rho(f; T) = \sup_{\mathbb{Q} \in \mathcal{M}_T^e} \left( \mathbb{E}^{\mathbb{Q}}[-f] - \alpha(\mathbb{Q}) \right), \quad (2.6)$$

for some convex penalty function  $\alpha : \mathcal{M}_T^e \rightarrow [0, \infty]$ . See Theorem 17, p. 445 in [11] for the proof in the discrete-time case. The proof in our setting is similar.

### 3. MATURITY-INDEPENDENT RISK MEASURES

**3.1. The need for maturity independence.** The classical notion of a convex risk measure, as well as its replication-invariant specialization, is inextricably linked to a specific maturity date with respect to which risk measurement is taking place while ignoring all other time instances. On the other hand, a fundamental property of financial markets is that they facilitate transfers of wealth among different time points as well as between different states of the world. The notion of replication invariance, introduced above, abstracts the latter property and ties it to the decision-theoretic notion of a convex risk measure. The former property, however, has not yet been incorporated into the risk measurement framework in the same manner in the existing literature. One of the goals herein is to do exactly this. We, then, pose and address the following question:

*“Is there a class of risk measures that are not constructed in reference to a specific time instance and can be, thus, used to measure the risk content of claims of all (arbitrary) maturities?”*

Equivalently, we wish to avoid the case when two versions of the same risk measure (differing only on the choice of the maturity date) give different risk values to the same contingent claim<sup>1</sup>.

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<sup>1</sup> One could object to the above reasoning by pointing out that different maturities *should* give rise to different risk assessments due to the effect of time impatience. In response, we take a view that the market is efficient in the sense that all time impatience is already incorporated in the investment possibilities present in it. More specifically, we remind the reader that the assumption that  $S^0 \equiv 1$  effectively means that all contingent claims are quoted in terms of time-0 currency. One can easily extend the theory presented here

Before we proceed with formal definitions, let us recall some of the fundamental properties of the arbitrage-free pricing (“Black-Scholes”) functional,  $\rho_{BS}$ , in the context of a complete financial market. For a “regular-enough” contingent claim  $f$ , the value  $\rho_{BS}(f)$  is defined as the capital needed at inscription to replicate it perfectly. The functional  $\rho_{BS}$  satisfies the axioms of convex risk measures and is replication-invariant. Moreover, it is per se unaffected by the expiration date of the generic claim  $f$ .

When markets are incomplete, a much more interesting set of phenomena occurs, as there is no canonical (“Black-Scholes”) pricing mechanism. We shall see that, interestingly, some traditional and widely used risk measures are *not* maturity-independent. In other words, under these measures, indifference prices of the same contingent claim, but calculated in terms of two distinct maturities will, in general, *differ*.

**3.2. Definition of maturity independence.** Let  $\mathbb{L}$  denote the set of all bounded random variables with finite maturities, i.e.,

$$\mathbb{L} = \cup_{t \geq 0} \mathbb{L}^\infty(\mathcal{F}_t).$$

The set  $\mathbb{L}$  will serve as a natural domain for the class of risk measures we propose in the sequel. Note that  $\mathbb{L}$  contains all  $\mathcal{F}_t$ -measurable bounded contingent claims, for all times  $t \geq 0$ , but it avoids the (potentially pathological) cases of random variables in  $\mathbb{L}^\infty(\mathcal{F}_\tau)$ , where  $\tau$  is a finite, but possibly unbounded, stopping time.

We are now ready to define the class of maturity-independent risk measures. With a slight abuse of notation, we still use the symbol  $\rho$ . In contrast to their maturity-dependent counterparts  $\rho(\cdot; T)$ , however, all maturity-specific notation has vanished.

**Definition 3.1.** A functional  $\rho : \mathbb{L} \rightarrow \mathbb{R}$  is called a **maturity-independent convex risk measure** if it has the following properties for all  $f, g \in \mathbb{L}$ , and  $\lambda \in [0, 1]$ :

- (1)  $\rho(f) \leq 0, \forall f \geq 0,$  (anti-positivity)
- (2)  $\rho(\lambda f + (1 - \lambda)g) \leq \lambda \rho(f) + (1 - \lambda)\rho(g),$  (convexity)
- (3)  $\rho(f - m) = \rho(f) + m, \forall m \in \mathbb{R},$  and (cash-translativity)
- (4) for all  $t \geq 0$ , and  $\pi \in \mathcal{A}_{\text{bd}}, \rho(f + \int_0^t \pi_s dS_s) = \rho(f).$  (replication and maturity independence)

We note that the properties which differentiate the maturity-independent risk measures from the existing notions are the *choice of the domain*  $\mathbb{L}$  on the one hand, and the validity of axiom (4) for *all* maturities  $t \geq 0$  on the other.

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to the more general case where the time-value of money is modeled explicitly. We feel, however, that such a generalization would only obscure the central issue herein and render the present paper less accessible.



**3.3. Motivational examples.** We start off our investigation of maturity-independent risk measures by giving three examples - one of an extremal such risk measure, one of a class of maturity-independent risk measures for closed markets, and one in which the maturity independence property fails.

**3.3.1. Super-hedging prices.** The simplest example of a maturity-independent risk measure is the super-hedging price function  $\hat{\rho} : \mathbb{L} \rightarrow \mathbb{R}$  given by

$$\hat{\rho}(f) = \inf \left\{ m \in \mathbb{R} : \exists \pi \in \mathcal{A}_{\text{bd}}, m + \int_0^\infty \pi_s dS_s \geq f, \text{ a.s.} \right\}.$$

It is easy to see that it satisfies all axioms in Definition 3.1. As in the maturity-dependent case,  $\hat{\rho}$  has the extremal property  $\hat{\rho}(f) \geq \rho(f)$ , for any  $f \in \mathbb{L}$  and any maturity-independent risk measure  $\rho$ .

**3.3.2. The case of closed markets.** The dual characterization (2.6) of replication-invariant risk measures for finite maturities can be used to construct maturity-independent risk measures when the market model is closed (see Definition 2.1 and paragraph 2.1.4 for notation and terminology). Indeed, let  $\alpha : \mathcal{M}_\infty^e \rightarrow [0, \infty]$  be a proper function (i.e., satisfying  $\alpha(\mathbb{Q}) < \infty$ , for at least one  $\mathbb{Q} \in \mathcal{M}_\infty^e$ .) It is not difficult to check that the functional  $\rho : \mathbb{L} \rightarrow \mathbb{R}$ , defined by

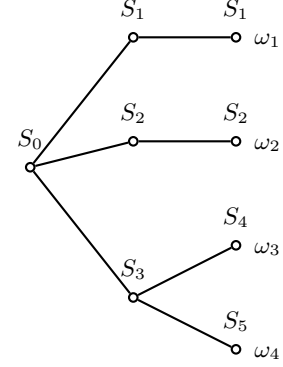
$$\rho(f) = \sup_{\mathbb{Q} \in \mathcal{M}_\infty^e} \left( \mathbb{E}^{\mathbb{Q}}[-f] - \alpha(\mathbb{Q}) \right),$$

is a maturity-independent risk measure. We have already seen that many market models used in practice are not closed. The natural construction used above will clearly not be applicable in those cases and, thus, an entirely different approach will be needed.

**3.3.3. Risk measures lacking maturity independence.** It is tempting to assume that a maturity-independent risk measure  $\rho$  can always be constructed by identifying a maturity date  $t$  associated with a contingent claim  $f$ , and setting  $\rho(f) = \rho(f; t)$ , for some replication-invariant risk measure  $\rho(\cdot; t)$ . As shown in the following two examples, this construction will not always be possible even if we restrict our attention to the well-explored class of entropic risk measures. Both examples are based on the entropic risk measure (see Example 2.6 (2)). Note that the first example is not entirely set in the framework described in section 2, but the reader will easily make all required (formal) modifications.

**a) A non-compliance example on a finite probability**

**space.** We present a simple two-period example in which entropic risk measurement gives different results for the same, time-1-measurable contingent claim  $f$ , when considered at time 1 and time 2. The market structure is described by the simple tree in Figure 1, where the (physical) probability of each of the branches leaving the initial node is  $\frac{1}{3}$ , and the conditional probabilities of the two contingencies (leading to  $S_4$  and  $S_5$ ) after the node  $S_3$  are equal to  $\frac{1}{3}$  and  $\frac{2}{3}$ , respectively. One can implement the described situation on a 4-element probability space  $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ , as in Figure 1, with  $\mathbb{P}[\omega_1] = \mathbb{P}[\omega_2] = 1/3$ ,  $\mathbb{P}[\omega_3] = 1/9$  and  $\mathbb{P}[\omega_4] = 2/9$ .



**Figure 1.** The market tree

There are two financial instruments: a riskless bond  $S^0 \equiv 1$ , and a stock  $S = S^1$  whose price is denoted by  $S_0, \dots, S_5$  for various nodes of the information tree, such that the following relations hold:

$$S_0 = S_2, \quad S_2 = \frac{1}{2}(S_1 + S_3), \quad S_1 \neq S_3, \quad S_3 = \frac{1}{2}(S_4 + S_5), \quad S_4 \neq S_5.$$

This implies, in particular, that the market is arbitrage-free, and, due to its incompleteness, the set of equivalent martingale measures is larger than just a singleton. Next, we consider a family  $\{f_a\}_{a>0}$  of contingent claims defined by

$$f_a(\omega) = \begin{cases} 0, & \omega = \omega_1, \omega_2, \\ a, & \omega = \omega_3, \omega_4. \end{cases}$$

We are going to compare  $\rho(f_a; 1)$  and  $\rho(f_a; 2)$  where  $\rho(f_a; t)$ ,  $t = 1, 2$ , is the value of the entropic risk measure (as defined in (2.4) above) of the contingent claim  $f_a$ , seen as time- $t$  random variable (note that  $f_a$  is  $\mathcal{F}_1$ -measurable, for all  $a$ ).

Let us first focus on  $\rho(f; 2)$ . The set of all martingale measures is given by  $\mathcal{M} = \{\mathbb{Q}^\nu : \nu \in (-\frac{1}{6}, \frac{1}{3})\}$ , where

$$\mathbb{Q}^\nu(\omega) = \begin{cases} \frac{1}{3} - \nu, & \omega = \omega_1, \\ \frac{1}{3} + 2\nu, & \omega = \omega_2, \\ \frac{1}{2}(\frac{1}{3} - \nu), & \omega = \omega_3, \omega_4. \end{cases}$$

By a finite-dimensional analogue of (2.5), we have

$$\rho(f_a; 2) = \sup_{\nu \in (-1/6, 1/3)} \left( \mathbb{E}^{\mathbb{Q}^\nu}[-f_a] - h_2(\nu) \right) = \sup_{\nu \in (-1/6, 1/3)} \left( -a(1/3 - \nu) - h_2(\nu) \right), \quad (3.1)$$

where, as one can easily check, the relative-entropy function  $h_2$  is given by

$$h_2(\nu) = \bar{h}_2(\nu) - \inf_{\mu} \bar{h}_2(\mu),$$

where

$$\bar{h}_2(\nu) = \frac{\mathbb{Q}^\nu[\omega_1]}{\mathbb{P}[\omega_1]} \ln \left( \frac{\mathbb{Q}^\nu[\omega_1]}{\mathbb{P}[\omega_1]} \right) + \frac{\mathbb{Q}^\nu[\omega_2]}{\mathbb{P}[\omega_2]} \ln \left( \frac{\mathbb{Q}^\nu[\omega_2]}{\mathbb{P}[\omega_2]} \right) + \frac{\mathbb{Q}^\nu[\omega_3]}{\mathbb{P}[\omega_3]} \ln \left( \frac{\mathbb{Q}^\nu[\omega_3]}{\mathbb{P}[\omega_3]} \right) + \frac{\mathbb{Q}^\nu[\omega_4]}{\mathbb{P}[\omega_4]} \ln \left( \frac{\mathbb{Q}^\nu[\omega_4]}{\mathbb{P}[\omega_4]} \right).$$

Similarly,

$$\rho(f_a; 1) = \sup_{\nu \in (-1/6, 1/3)} \left( \mathbb{E}^{\mathbb{Q}^\nu}[-f_a] - h_1(\nu) \right) = \sup_{\nu \in (-1/6, 1/3)} \left( -a(1/3 - \nu) - h_1(\nu) \right), \quad (3.2)$$

where the function  $h_1$  is given by  $h_1(\nu) = \bar{h}_1(\nu) - \inf_\nu \bar{h}_1(\nu)$ , with

$$\bar{h}_1(\nu) = \mathbb{Q}^\nu[\omega_1] \ln \left( \frac{\mathbb{Q}^\nu[\omega_1]}{\mathbb{P}[\omega_1]} \right) + \mathbb{Q}^\nu[\omega_2] \ln \left( \frac{\mathbb{Q}^\nu[\omega_2]}{\mathbb{P}[\omega_2]} \right) + (\mathbb{Q}^\nu[\omega_3] + \mathbb{Q}^\nu[\omega_4]) \ln \left( \frac{\mathbb{Q}^\nu[\omega_3] + \mathbb{Q}^\nu[\omega_4]}{\mathbb{P}[\omega_3] + \mathbb{P}[\omega_4]} \right).$$

The expressions (3.1) and (3.2) can be seen as the Legendre-Fenchel transforms of the translated entropy functions  $h_2(1/3 - \nu)$  and  $h_1(1/3 - \nu)$ . Therefore, by the bijectivity of these transforms and the convexity of the functions  $h_1$  and  $h_2$ , the equality  $\rho(f_a; 1) = \rho(f_a; 2)$ , for all  $a > 0$ , would imply that  $h_1 = h_2$ . It is now a matter of a straightforward computation to show that that is, in fact, not the case. Thus, the two values do not coincide, i.e., for at least one  $a > 0$ ,

$$\rho(f_a; 1) \neq \rho(f_a; 2).$$

#### b) A non-compliance example in a diffusion market model.

We consider a financial market as in section 2, with  $k = 1$  (one risky asset) and  $d = 2$  (two driving Brownian motions). It will be enough to consider a stock price process with stochastic volatility of the form

$$\begin{aligned} dS_s &= S_s(\mu ds + \sigma(\tilde{B}_s) dW_s^1), \\ d\tilde{B}_s &= dB_s, \end{aligned} \quad (3.3)$$

$s \geq 0$ , on an augmented filtration generated by two independent Brownian motions  $W^1$  and  $W^2$ , where  $B = \rho W^1 + \sqrt{1 - \rho^2} W^2$  is a Brownian motion correlated with  $W^1$ , with the correlation coefficient  $\rho \in (0, 1)$ . It will be convenient to introduce the market price of risk  $\lambda(y) = \mu/\sigma(y)$ , assuming throughout that  $\lambda : \mathbb{R} \rightarrow (0, \infty)$  is a strictly increasing  $C^1$ -function with range of the form  $(\varepsilon, M)$  for some constants  $0 < \varepsilon < M < \infty$ . The trading starts at time  $t$ , after which two maturities  $T, \bar{T}$ , with  $T < \bar{T}$ , are chosen.

Let  $C_T = -B_T$  model the payoff of a contingent claim which is, clearly, nonreplicable. The value of the time- $t$  entropic ( $\gamma = 1$ ) risk measure  $\rho_t(C_T; T)$  equals the indifference price  $\nu_t(-C_T; T)$  of the claim  $B_T$  measured on the trading horizon  $[t, T]$ . According to [31],  $\rho_t(C_T; T)$  admits a representation in terms of a solution to a partial differential equation. More precisely, taking into account the fact that neither the payoff  $C_T$  nor the dynamics of the volatility depend on the stock price, we have  $\rho_t(C_T; T) = p(t, -\tilde{B}_t)$ , a.s., where the function  $p : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a classical solution of the quasilinear equation

$$\begin{cases} p_t + \mathcal{L}^f p + \frac{1}{2}(1 - \rho^2)p_y^2 = 0 \\ p(T, y) = y, \end{cases} \quad (3.4)$$

where  $\mathcal{L}^f p = \frac{1}{2} p_{yy} + (f_y/f - \rho\lambda(y))p_y$ . The function  $f : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is the unique solution to the linear problem

$$\begin{cases} f_t + \mathcal{A}f = 0 \\ f(T, y) = 1, \end{cases} \quad (3.5)$$

where  $\mathcal{A}f = \frac{1}{2} f_{yy} - \rho\lambda(y)f_y - \frac{1}{2}(1 - \rho^2)\lambda^2(y)f$ . Standard arguments show that  $f$  is of class  $C^{1,3}$  and admits a representation in the manner of Feynman and Kac as

$$f(t, y) = \mathbb{E}[e^{\int_t^T \frac{(1-\rho^2)}{2} \lambda^2(Y_s) ds} | Y_t = y], \quad (t, y) \in [0, T] \times \mathbb{R}, \quad (3.6)$$

where  $\{Y_s\}_{s \in [t, \infty)}$  is the unique strong solution to  $dY_s = dB_s - \rho\lambda(Y_s)ds$ ,  $Y_t = y$ . In particular, there exists a constant  $C > 1$  such that  $1 \leq f(t, y) \leq C$ , for  $(t, y) \in [0, T] \times \mathbb{R}$ .

Similarly, the indifference price  $\nu_t(-C_T; \bar{T})$  (which equals the value  $\rho_t(C_T; \bar{T})$  of the maturity- $\bar{T}$  entropic risk measure  $\rho_t(\cdot; \bar{T})$  applied to the same contingent claim, only on the longer horizon  $[0, \bar{T}]$ ,  $\bar{T} > T$ ) can be represented via  $\bar{p}(t, y)$ , where  $\bar{p}$  solves

$$\begin{cases} \bar{p}_t + \mathcal{L}^{\bar{f}} \bar{p} + \frac{1}{2}(1 - \rho^2)\bar{p}_y^2 = 0 \\ \bar{p}(T, y) = y. \end{cases} \quad (3.7)$$

Herein,  $\mathcal{L}^{\bar{f}}$  is given as in (3.5) with  $f$  replaced by the function  $\bar{f}$  which solves

$$\begin{cases} \bar{f}_t + \mathcal{A}\bar{f} = 0, \\ \bar{f}(\bar{T}, y) = 1. \end{cases} \quad (3.8)$$

Just like  $f$ , the function  $\bar{f}$  admits a representation analogous to (3.6) and a uniform bound  $1 \leq \bar{f}(t, y) \leq \bar{C}$ , for  $(t, y) \in [0, T] \times \mathbb{R}$ .

The goal of this example is to show that the indifference prices  $\nu(B_T; T)$  and  $\nu(B_T; \bar{T})$ , or, equivalently, the entropic risk measures  $\rho_t(C_T; T)$  and  $\rho_t(C_T; \bar{T})$ , do *not* always coincide, i.e., that  $p(t, y)$  and  $\bar{p}(t, y)$  differ for at least one choice of  $(t, y) \in [0, T] \times \mathbb{R}$ . We start with an auxiliary result, namely,

$$\frac{f_y(T, y)}{f(T, y)} \neq \frac{\bar{f}_y(T, y)}{\bar{f}(T, y)}, \quad \text{for each } y \in \mathbb{R}. \quad (3.9)$$

In order to establish (3.9), we note that the function  $g : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ , defined by  $g = f_y$ , is a classical solution to

$$\begin{cases} g_t + \mathcal{B}g = 0 \\ g(T, y) = 0, \end{cases} \quad (3.10)$$

where

$$\mathcal{B}g = \frac{1}{2} g_{yy} - \rho\lambda(y)g_y - A(y)g - B(t, y),$$

with  $A(y) = \rho\lambda'(y) + \frac{1}{2}(1 - \rho^2)\lambda^2(y)$  and  $B(t, y) = (1 - \rho^2)\lambda(y)\lambda'(y)f(t, y)$ .

Thanks to the assumptions placed on  $\rho$  and  $\lambda$ , and the positivity of  $f$ , we have

$$A(y) > 0 \text{ and } B(t, y) > 0, \text{ for all } (t, y) \in [0, T] \times \mathbb{R}. \quad (3.11)$$

The function  $\bar{g} = \bar{f}_y$  is defined in an analogous fashion (only on the larger domain  $[0, \bar{T}] \times \mathbb{R}$ ) and a similar set of properties can be derived. Since  $f_y(T, y) = 0$  for all  $y \in \mathbb{R}$ , it will be enough to show that  $\bar{f}_y(T, y) > 0$  for all  $y \in \mathbb{R}$ . This follows immediately from the strict inequalities in (3.11) and the Feynman-Kac representation

$$\bar{g}(T, y) = \bar{f}_y(T, y) = \mathbb{E}\left[\int_T^{\bar{T}} B(t, Y_t) e^{\int_t^{\bar{T}} A(Y_s) ds} dt | Y_T = y\right], \quad y \in \mathbb{R}. \quad (3.12)$$

Having established (3.9), we conclude that, thanks to the smoothness of the functions  $f$  and  $\bar{f}$ , the operators  $\mathcal{L}^f$  and  $\mathcal{L}^{\bar{f}}$  differ in the  $\frac{\partial}{\partial y}$ -coefficient in some open neighbourhood  $\mathcal{N}$  of the line  $\{T\} \times \mathbb{R}$  in  $[0, T] \times \mathbb{R}$ . Assuming that  $\bar{p}$  and  $p$  coincide in  $\mathcal{N}$ , subtracting the equations (3.4) and (3.7) yields

$$\left(\frac{f_y}{f}(t, y) - \frac{\bar{f}_y}{\bar{f}}(t, y)\right) \bar{p}_y(t, y) = 0, \quad \text{for } (t, y) \in \mathcal{N}. \quad (3.13)$$

Equation (3.9) now implies that  $\bar{p}_y = 0$  on  $\mathcal{N}$ , which is clearly in contradiction with the terminal condition  $\bar{p}(T, y) = y$ ,  $y \in \mathbb{R}$ . Therefore, there exists  $(t, y) \in \mathcal{N} \setminus \{T\} \times \mathbb{R} \subseteq [0, T) \times \mathbb{R}$  such that  $p(t, y) \neq \bar{p}(t, y)$ .

#### 4. FORWARD ENTROPIC RISK MEASURES (FERM)

In the previous section, we saw three examples of risk measures and their dependence on the specific choice of the maturity date. In particular, we pointed out that the super-hedging risk measure in 3.3.1, as well as the ones constructed in 3.3.2, for the class of closed markets, are maturity-independent. However, both these classes are rather restrictive. Indeed, the one associated with super-hedging is prohibitively conservative, while the other requires the rather stringent assumption of market closedness.

In this section, we introduce a new family of convex risk measures that have the **maturity independence** property and, at the same time, are applicable to a wide range of settings. Their construction is based on the idea mentioned in the introductory paragraph of Subsection 3.3.3, but avoids the pitfalls responsible for the failure of examples a) and b) following it.

The risk measures we are going to introduce are closely related to indifference prices. The novelty of the approach is that the underlying risk preference functionals are not tied down to a specific maturity, as it has been the case in the standard expected utility formulation. Rather, they can be seen as specified at initiation and subsequently “generated” across all times. This approach was proposed by the first author and M. Musiela (see [24, 25, 26, 27, 28]) and is briefly reviewed below.

**4.1. Forward exponential performances.** The notion of a forward performance process has arisen from the search for ways to measure the performance of investment strategies across *all* times in  $[0, \infty)$ . In order to produce a nontrivial such object, we look for a random field  $U = U_t(\omega, x)$  defined for *all* times  $t \geq 0$  and parametrized by a wealth argument  $x$

such that the mapping  $x \mapsto U_t(\omega, x)$  admits the classical properties of utility functions. More precisely, we have the following definition:

**Definition 4.1.** A mapping  $U : [0, \infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is called a **performance random field** if

- (1) for each  $(t, \omega) \in [0, \infty) \times \Omega$ , the mapping  $x \mapsto U_t(x, \omega)$  defines a utility function: it is strictly concave, strictly increasing, continuously differentiable and satisfies the Inada conditions  $\lim_{x \rightarrow \infty} U'(x) = 0$  and  $\lim_{x \rightarrow -\infty} U'(x) = +\infty$ ,
- (2)  $U(\cdot, \cdot)$  is measurable with respect to the product of the progressive  $\sigma$ -algebra on  $\Omega \times [0, \infty)$  and the Borel  $\sigma$ -algebra on  $\mathbb{R}$ , and
- (3)  $\mathbb{E} |U_t(x)| < \infty$ , for all  $(t, x) \in [0, \infty) \times \mathbb{R}$ .

*Remark 4.2.*

- (1) The last requirement in Definition 4.1 implies, in particular, that  $\mathbb{E} |U_t(\xi)| < \infty$ , for all random variables  $\xi \in \mathbb{L}^\infty$ .
- (2) It is possible to construct a parallel theory where the performance functions  $U_t(\omega, \cdot)$  are defined on the positive semi-axis  $(0, \infty)$ . We choose the domain  $\mathbb{R}$  for the wealth argument  $x$  because it leads to a slightly simpler analysis, and because the examples to follow will be based on the exponential function.

On an arbitrary trading horizon, say  $[s, t]$ ,  $0 \leq s < t < \infty$ , the investor whose preferences are described by the random field  $U$  seeks to maximize the expected investment performance:

$$V_s(x) = \operatorname{esssup}_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E}[U_t(X_t^{x, \pi}) | \mathcal{F}_s], \quad 0 \leq s \leq t. \quad (4.1)$$

Herein,  $X^{x, \pi}$  denotes the investor's wealth process,  $x \in \mathbb{R}$  the investor's initial wealth at time  $s$ , and  $\pi$  a generic investment strategy belonging to  $\mathcal{A}_{\text{bd}}$  (the set of admissible policies introduced in Subsection 2.1.2.) To concentrate on the new notions, we abstract throughout from control and state constraints, as well as the most general specification of admissibility requirements.

It has been argued in [28] that the class of performance random fields with the additional property

$$V_t(x) = U_t(x), \quad \text{a.s.} \quad \forall t \in [0, \infty), \quad x \in \mathbb{R}, \quad (4.2)$$

possesses several desirable properties and gives rise to an analytically tractable theory.

**Definition 4.3.** A random field  $U$  satisfying (4.2), where  $V$  is defined by (4.1), is called **self-generating**.

*Remark 4.4.* We remind the reader that a classical example of a self-generating performance random field (albeit *only on the finite horizon*  $[0, T]$ ) is the traditional value function, defined as

$$U_t(x) = \operatorname{esssup}_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E}[U_T(X_T^{x, \pi}) | \mathcal{F}_t], \quad t \in [0, T], \quad x \in \mathbb{R},$$

where  $T$  is a prespecified maturity beyond which no investment activity is measured, and  $U_T(\cdot, \cdot) : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$  is a classical (state-dependent) utility function (see, for example, [19, 21, 22 and 32]). When the horizon is infinite, such a construction will not produce any results. Indeed, there is no appropriate time for the final datum to be given.

What (4.1) and (4.2) tell us is that (under additional regularity conditions) the sought-after criterion (performance random field)  $U$  must have the property that the stochastic process  $U_t(X_t^{x,\pi})$  is a supermartingale for an arbitrary control  $\pi \in \mathcal{A}_{\text{bd}}$  and becomes “closer and closer” to a martingale as the controls get “better and better”. In the case when the class of control problems (4.1) actually admits an optimizer  $\pi^* \in \mathcal{A}_{\text{bd}}$  (or in some larger, appropriately chosen, class), the composition  $U_t(X_t^{x,\pi^*})$  becomes a martingale.

In the traditional framework, as already mentioned in Remark 4.4, the datum (terminal utility) is assigned at some fixed future time  $T$ . Alternatively, in the case of an infinite time horizon, it is more natural to think of the datum  $u_0 : \mathbb{R} \rightarrow \mathbb{R}$  as being assigned at time  $t = 0$ , and a self-generating performance random field  $U_t$  chosen so that  $U_0(x) = u_0(x)$ . It is because of this interpretation that the self-generating performance random fields may, also, be referred to as **forward performances**.

The notion of forward performance processes was first developed for binomial models in [24] and [25] and later generalized to diffusion models with a stochastic factor ([26]) and, more recently, to models of Itô asset price dynamics (see, among others, [26] and [28], as well as [4]). A related stochastic optimization problem that allows for semimartingale price processes and random horizons can be found in [8]. A similar notion of utilities without horizon preference was developed in [18]; therein, asset prices are taken to be lognormal, leading to deterministic forward solutions.

While traditional performance random fields on finite horizons are straightforward to construct and characterize, producing a “forward” performance random field on  $[0, \infty)$  from a given initial datum  $u_0$  is considerably more difficult. Several examples of such a construction, all based on the exponential initial datum, are given in the following subsection. These random fields are the most important building blocks for the class of maturity-independent risk measures presented in subsection 4.2, below.

**Definition 4.5.** A performance random field  $U$  is called a **forward exponential performance** if

- a) it is self-generating, and
- b) there exists a constant  $\gamma > 0$ , such that

$$U_0(x) = -e^{-\gamma x}, \quad x \in \mathbb{R}. \quad (4.3)$$

The construction presented below can be found in [27]. The assumptions and definitions from section 2 will be used in the sequel without explicit mention.

**Theorem 4.6** (Theorem 4 in [27]). *Let  $(Y_t)_{t \in [0, \infty)}$ ,  $(Z_t)_{t \in [0, \infty)}$  be two continuous processes solving*

$$dY_t = Y_t \delta_t (\lambda_t dt + dW_t) \quad (4.4)$$

and

$$dZ_t = Z_t \phi_t dW_t, \quad (4.5)$$

with  $Y_0 = 1/\gamma > 0$ ,  $Z_0 = 1$ , for a fixed, but arbitrary  $k$ -dimensional coefficient processes  $(\delta_t)_{t \in [0, \infty)}$  and  $(\phi_t)_{t \in [0, \infty)}$  which are assumed to be  $\mathbb{F}$ -adapted, and with  $\delta$  satisfying  $\sigma_t \sigma_t^+ \delta_t = \delta_t$ , for all  $t \geq 0$ , a.s. We, also, assume that  $\delta$  and  $\phi$  are regular enough for the integrals in (4.4) and (4.5) to be well defined, and that, when restricted to any finite interval  $[0, t]$ , the process  $Z$  is a positive martingale, and  $Y$  is uniformly bounded from above and away from zero.

Let the process  $(A_t)_{t \in [0, \infty)}$  be defined as

$$A_t = \int_0^t \|\sigma_s \sigma_s^+ (\lambda_s + \phi_s) - \delta_s\|^2 ds. \quad (4.6)$$

Then, the random field  $U$ , given by

$$U_t(x; \omega) = -Z_t \exp \left( -\frac{x}{Y_t} + \frac{A_t}{2} \right), \quad (4.7)$$

is a forward exponential performance. In particular, for  $0 \leq s \leq t$  and  $\xi \in \mathbb{L}^\infty(\mathcal{F}_s)$ , we have

$$U_s(\xi) = \text{esssup}_{\pi \in \mathcal{A}_{bd}} \mathbb{E} \left[ U_t \left( \xi + \int_s^t \pi_u dS_u \right) \middle| \mathcal{F}_s \right], \text{ a.s.} \quad (4.8)$$

*Remark 4.7.* In (4.7) above, one can give a natural financial interpretation to the processes  $Y$  (which normalizes the wealth argument) and  $Z$  (which appears as a multiplicative factor). One might think of  $Y$  as a benchmark (or a numéraire) in relation to which we wish to measure the performance of our investment strategies. The values of the process  $Z$ , on the other hand, can be thought of as Radon-Nikodym derivatives of the investor's subjective probability measure with respect to the measure  $\mathbb{P}$ .

**4.2. Forward entropic risk measures.** We are now ready to introduce the *forward entropic risk measures* (FERM). We start with an auxiliary object, denoted by  $\rho(C; t)$ .

**Definition 4.8.** Let  $U$  be the forward exponential performance defined in (4.7), and let  $t \geq 0$  be arbitrary, but fixed. For a contingent claim written at time  $s = 0$  and yielding a payoff  $C \in \mathbb{L}^\infty(\mathcal{F}_t)$ , we define  $\rho(C; t) \in \mathbb{R}$  as the unique solution of

$$\sup_{\pi \in \mathcal{A}_{bd}} \mathbb{E} \left[ U_t \left( x + \int_0^t \pi_s dS_s \right) \right] = \sup_{\pi \in \mathcal{A}_{bd}} \mathbb{E} \left[ U_t \left( x + \rho(C; t) + C + \int_0^t \pi_s dS_s \right) \right], \quad \forall x \in \mathbb{R}. \quad (4.9)$$

The mapping  $\rho(\cdot; t) : \mathbb{L}^\infty(\mathcal{F}_t) \rightarrow \mathbb{R}$  is called the  **$t$ -normalized forward entropic measure**.



One can, readily, check that the equation (4.9) indeed admits a unique solution (independent of the initial wealth  $x$ ), so that the  $t$ -normalized forward entropic measures are well defined. The reader can convince him-/herself of the validity of the following result:

**Proposition 4.9.** *The  $t$ -normalized forward entropic risk measures are replication-invariant convex risk measures on  $\mathbb{L}^\infty(\mathcal{F}_t)$ , for each  $t \geq 0$ .*

The *fundamental* property in which forward entropic risk measures differ from a generic replication-invariant risk measure (see examples in Subsection 3.3.2) is the following:

**Proposition 4.10.** *For  $0 \leq s < t < \infty$ , and  $C^{(s)} \in \mathbb{L}^\infty(\mathcal{F}_s)$ , consider the  $s$ - and  $t$ -normalized forward entropic measures  $\rho(C^{(s)}; s)$  and  $\rho(C^{(s)}; t)$  applied to the contingent claim  $C^{(s)}$ . Then,*

$$\rho(C^{(s)}; s) = \rho(C^{(s)}; t). \quad (4.10)$$

More generally, for  $C^{(r)} \in \mathbb{L}^\infty(\mathcal{F}_r)$ , where  $0 \leq r < s < t < \infty$ , we have

$$\rho(C^{(r)}; s) = \rho(C^{(r)}; t). \quad (4.11)$$

*Proof.* We are only going to establish (4.10) since (4.11) follows from similar arguments. To this end, note that a self-financing policy  $\pi \in \mathcal{A}_{bd}$  if and only if  $\pi \mathbf{1}_{[0,t]} \in \mathcal{A}_{bd}$  and  $\pi \mathbf{1}_{(t,\infty)} \in \mathcal{A}_{bd}$ . Using Definition 4.8 at  $x = 0$ , we obtain

$$\begin{aligned} U_0(0) &= \sup_{\pi \in \mathcal{A}_{bd}} \mathbb{E}[U_t(\rho(C^{(s)}; t) + C^{(s)} + \int_0^t \pi_u dS_u)] \\ &= \sup_{\pi, \pi' \in \mathcal{A}_{bd}} \mathbb{E} \left[ \mathbb{E}[U_t(\rho(C^{(s)}; t) + C^{(s)} + \int_0^s \pi_u dS_u + \int_s^t \pi'_u dS_u) | \mathcal{F}_s] \right] \\ &= \sup_{\pi \in \mathcal{A}_{bd}} \mathbb{E} \left[ \text{esssup}_{\pi' \in \mathcal{A}_{bd}} \mathbb{E}[U_t(\rho(C^{(s)}; t) + C^{(s)} + \int_0^s \pi_u dS_u + \int_s^t \pi'_u dS_u) | \mathcal{F}_s] \right] \\ &= \sup_{\pi \in \mathcal{A}_{bd}} \mathbb{E} \left[ U_s(\rho(C^{(s)}; t) + C^{(s)} + \int_0^s \pi_u dS_u) \right], \end{aligned}$$

where we used the semigroup property (4.8) of  $U$  and the fact that the random variable  $\rho(C^{(s)}; t) + C^{(s)} + \int_0^s \pi_u dS_u$  is an element of  $\mathbb{L}^\infty(\mathcal{F}_s)$ , for all  $\pi \in \mathcal{A}_{bd}$ . We compare the obtained expression with the defining equation (4.9) to conclude that  $\rho(C^{(s)}; t) = \rho(C^{(s)}; s)$ .  $\square$

We are now ready to define the forward entropic risk measures:

**Definition 4.11.** For  $C \in \mathbb{L}$ , define the **earliest maturity**  $t_C \in [0, \infty)$  of  $C$  as

$$t_C = \inf \{t \geq 0 : C \in \mathcal{F}_t\}. \quad (4.12)$$

The **forward entropic risk measure**  $\nu : \mathbb{L} \rightarrow \mathbb{R}$  is defined as

$$\rho(C) = \rho(C; t_C), \quad (4.13)$$

where  $\rho(C; t_C)$  is the value of the  $t_C$ -normalized forward entropic risk measure, defined in (4.9), applied to the contingent claim  $C$ .

The focal point of the present section is the following theorem:

**Theorem 4.12.** *The mapping  $\rho : \mathbb{L} \rightarrow \mathbb{R}$  is a maturity-independent risk measure.*

*Proof.* We need to verify the axioms (1)-(4) of Definition 3.1. Axioms (1) and (3) follow directly from elementary properties of the  $t$ -normalized forward risk measures. To show axiom (2) we take  $\lambda \in (0, 1)$  and  $C_1, C_2 \in \mathbb{L}$ . Then, since  $\lambda C_1 + (1 - \lambda)C_2 \in \mathcal{F}_{\max(t_{C_1}, t_{C_2})}$ , we have  $\max(t_{C_1}, t_{C_2}) \geq t_{\lambda C_1 + (1 - \lambda)C_2}$ . Therefore,

$$\begin{aligned} \rho(\lambda C_1 + (1 - \lambda)C_2) &= \rho(\lambda C_1 + (1 - \lambda)C_2; t_{\lambda C_1 + (1 - \lambda)C_2}) \\ &= \rho(\lambda C_1 + (1 - \lambda)C_2; \max(t_{C_1}, t_{C_2})), \end{aligned}$$

where we used (4.10). Using property (4.10) and the fact that the  $t$ -forward entropic risk measures are convex risk measures, we get

$$\begin{aligned} \rho(\lambda C_1 + (1 - \lambda)C_2) &\leq \lambda \rho(C_1; \max(t_{C_1}, t_{C_2})) + (1 - \lambda) \rho(C_2; \max(t_{C_1}, t_{C_2})) \\ &= \lambda \rho(C_1; t_{C_1}) + (1 - \lambda) \rho(C_2; t_{C_2}) \\ &= \lambda \rho(C_1) + (1 - \lambda) \rho(C_2). \end{aligned}$$

It remains to check the replication and maturity independence axiom (4). To this end, we let  $\xi = \int_0^\infty \pi_u dS_u$  for some portfolio process  $\pi \in \mathcal{A}_{\text{bd}}$ . We need to show that

$$\rho(C + \xi) = \rho(C),$$

for any  $C \in \mathbb{L}$ . Observe that  $\max(t_C, t_\xi) \geq t_{C+\xi}$  and, therefore, by (4.10) and (4.13), we have

$$\rho(C + \xi) = \rho(C + \xi; t_{C+\xi}) = \rho(C + \xi; \max(t_C, t_\xi)).$$

On the other hand, Proposition 4.9, the form of  $\xi$  and (4.10) yield

$$\rho(C + \xi; \max(t_C, t_\xi)) = \rho(C; \max(t_C, t_\xi)) = \rho(C; t_C) = \rho(C),$$

establishing axiom (4). □

Next, we provide an explicit representation of the forward entropic risk measures.

**Theorem 4.13.** *Let  $Y, Z, A$  and  $U_t(\cdot)$  be as in Theorem 4.6. For  $C \in \mathbb{L}$ , its forward entropic risk measure is given by*

$$\rho(C) = \inf_{\pi \in \mathcal{A}_{\text{bd}}} \left( \frac{1}{\gamma} \ln \mathbb{E} \left[ -U_t \left( C + \int_0^t \pi_s dS_s \right) \right] \right), \quad \text{for any } t \geq t_C, \quad (4.14)$$

where  $t_C$  is defined in (4.12).

*Proof.* Equation (4.9) (with  $x = -\rho(C; t)$  and  $t \geq t_C$ ) and the property (4.8) of the random field  $U$ , yield that

$$-\exp(\gamma\rho(C)) = \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E}[U_t(C + \int_0^t \pi_s dS_s)], \text{ for any } t \geq t_C. \quad (4.15)$$

By (4.8), the right-hand side of (4.15) is independent of  $t$  for  $t \geq t_C$ .  $\square$

**4.3. Relationship with dynamic risk measures.** Before we present concrete examples of maturity-independent risk measures in section 5, let us briefly discuss their relationship with the dynamic risk measures (see the introduction for references). A family of mappings  $\rho_s(\cdot; t) : \mathbb{L}^\infty(\mathcal{F}_t) \rightarrow \mathbb{L}^\infty(\mathcal{F}_s)$ , where  $0 \leq s \leq t \leq T$ , with  $T \in [0, \infty]$ , is said to be a **dynamic (time-consistent) risk measure** if each  $\rho_s(\cdot; t)$  satisfies the analogues of the axioms of convex risk measures and the semi-group property

$$\rho_s(-\rho_t(f; u); t) = \rho_s(f; u), \quad 0 \leq s \leq t \leq u \leq T,$$

holds. Using a version of Definition 4.11 and Theorem 4.12, the reader can readily check that each replication-invariant dynamic risk measure defined on the whole positive semi-axis  $[0, \infty)$  (i.e., when  $T = \infty$ ) gives rise to a maturity-independent risk measure. Under certain conditions, the reverse construction can be carried out as well (details will be presented in [33]).

The philosophies of the two approaches are quite different, though. Perhaps the best way to illustrate this point is through the analogy with the expected utility theory. Dynamic risk measures correspond to the traditional utility framework where a system of decisions relating various maturity dates is interlaced together through a consistency criterion. The maturity-independent risk measures take the opposite point of view and correspond to forward performances. While the dynamic risk measures are natural in the case  $T < \infty$ , the maturity-independent risk measures fit well with infinite or un-prespecified maturities.

## 5. EXAMPLES

In this section, we provide two representative classes of forward entropic risk measures. First, we single out some of the special cases obtained when specific choices for the processes  $Z$  and  $Y$  (of Definition 4.6) are used in conjunction with Definition 4.11 of the forward entropic risk measures. Then, we illustrate the versatility of the general notion of maturity-independent risk measures by constructing an example in an incomplete binomial-type model. Even though the remainder of the paper is set in an Itô-process model framework, the latter example is not. The reader can easily translate all the relevant definitions and results to fit this model. Some background and technical details pertaining to this example can be found in [23].

**5.1. Itô-process-driven markets.** This example is set in a financial market described in section 2, with  $k = 1$  (one risky asset) and  $d = 2$  (two driving Brownian motions). Without loss of generality, we assume that  $\sigma_t^{12} \equiv 0$ , and  $\sigma_t = \sigma_t^{11} > 0$ , i.e., that the second Brownian motion does not drive the tradeable asset. In this case, we have  $\lambda_t = (\lambda_t^1, \lambda_t^2)$ , where  $\lambda_t^1 = \mu_t/\sigma_t$  and  $\lambda_t^2 = 0$ . Therefore, the stock-price process satisfies

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t^1),$$

on an augmented filtration generated by a 2-dimensional Brownian motion  $(W^1, W^2)$ . The processes  $Z, Y, A$  from Theorem 4.6 can be written as

$$dY_t = Y_t \delta_t (\lambda_t^1 dt + dW_t^1), \quad Y_0 = 1/\gamma > 0, \quad dZ_t = Z_t \phi_t dW_t^1, \quad Z_0 = 1, \quad (5.1)$$

and

$$A_t = \int_0^t (\lambda_s^1 + \phi_s - \delta_s)^2 ds, \quad A_0 = 0, \quad (5.2)$$

subject to a choice of two processes  $\phi$  and  $\delta$ , under the regularity conditions stated in Theorem 4.6.

**a)  $\phi \equiv \delta \equiv 0$ .** In this case,  $Z_t \equiv 1$ ,  $Y_t \equiv 1/\gamma$ ,  $A_t \equiv \int_0^t (\lambda_s^1)^2 ds$  and the random field  $U$  of (4.7) becomes

$$U_t(x) = -\exp(-\gamma x + \frac{A_t}{2}).$$

Using the indifference-pricing equation (4.9) and the self-generation property (4.8) of  $U_t$ , we deduce that for  $C \in \mathbb{L}$ , the value  $\rho(C)$  satisfies

$$-\exp(\gamma \rho(C)) = \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E} \left[ -\exp \left( -\gamma \left( C + \int_0^t \pi_s dS_s \right) + \frac{A_t}{2} \right) \right], \quad \text{for any } t \geq t_C.$$

On the other hand, the classical (exponential) indifference price,  $\nu(C - \frac{A_t}{2\gamma}; t)$ , of the contingent claim  $C - \frac{A_t}{2\gamma}$ , maturing at time  $t$ , satisfies

$$\begin{aligned} & \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E} \left[ -\exp \left( -\gamma \left( \nu(C - \frac{A_t}{2\gamma}; t) + \int_0^t \pi_s dS_s \right) \right) \right] = \\ & = \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E} \left[ -\exp \left( -\gamma \left( C - \frac{A_t}{2\gamma} + \int_0^t \pi_s dS_s \right) \right) \right]. \end{aligned}$$

With  $H_t = \ln \sup_{\pi \in \mathcal{A}_{\text{bd}}} \mathbb{E}[-\exp(-\gamma \int_0^t \pi_s dS_u)]$  (which will be recognized by the reader familiar with exponential utility maximization as the aggregate relative entropy), we now have

$$\rho(C) = -\nu(C - \frac{A_t}{2\gamma}; t) - \frac{1}{\gamma} H_t, \quad \text{for any } t \geq t_C. \quad (5.3)$$

**b)  $\delta \equiv 0$ .** Then  $Y_t \equiv 1/\gamma$ ,  $A_t \equiv \int_0^t (\lambda_s^1 + \phi_s)^2 ds$ , and the random field  $U$  of (4.7) takes the form

$$U_t(x) = -Z_t \exp(-\gamma x + \frac{A_t}{2}).$$

The risk measure  $\rho(C)$  can be represented as in (5.3) above, with one important difference. Specifically, the (physical) probability measure  $\mathbb{P}$  has to be replaced by the probability  $\tilde{\mathbb{P}}$  whose Radon-Nikodym derivative w.r.t.  $\mathbb{P}$  is given by  $Z_t$  on  $\mathcal{F}_t$ , for any  $t \geq 0$ .

We leave the discussion of further examples in this setting - in particular for the case  $\delta \neq 0$  - for the upcoming work of one of the authors [33].

**5.2. The binomial case.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which two sequences  $\{\xi_t\}_{t \in \mathbb{N}}$  and  $\{\eta_t\}_{t \in \mathbb{N}}$  of random variables are defined. The stochastic processes  $\{S_t\}_{t \in \mathbb{N}_0}$  and  $\{Y_t\}_{t \in \mathbb{N}_0}$  are defined, in turn, as follows:

$$S_t = \prod_{k=1}^t \xi_k, \quad Y_t = \prod_{k=1}^t \eta_k, \quad t \in \mathbb{N}, \quad S_0 = Y_0 = 1.$$

The process  $S$  models the evolution of a (traded) risky asset, and  $Y$  is a (non-traded) factor. We assume, for simplicity, that the agents are allowed to invest in a zero-interest riskless bond  $S^0 \equiv 1$ . The following two filtrations are naturally defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ :

$$\mathcal{F}_t^S = \sigma(S_0, S_1, \dots, S_t) = \sigma(\xi_1, \dots, \xi_t), \quad t \in \mathbb{N}_0, \quad \text{and}$$

$$\mathcal{F}_t = \sigma(S_0, Y_0, S_1, Y_1, \dots, S_t, Y_t) = \sigma(\xi_1, \dots, \xi_t, \eta_1, \dots, \eta_t), \quad t \in \mathbb{N}_0$$

We assume that for each  $t \in \mathbb{N}$ , there exist  $\xi_t^u, \xi_t^d, \eta_t^u, \eta_t^d \in \mathbb{R}$  with  $0 < \xi_t^d < 1 < \xi_t^u$  and  $0 < \eta_t^d < \eta_t^u$  such that  $\mathbb{P}[\xi_t = \xi_t^u | \mathcal{F}_{t-1}] = 1 - \mathbb{P}[\xi_t = \xi_t^d | \mathcal{F}_{t-1}] > 0$ , a.s., and  $\mathbb{P}[\eta_t = \eta_t^u] = 1 - \mathbb{P}[\eta_t = \eta_t^d]$ .

The agent starts with initial wealth  $x \in \mathbb{R}$ , and trades in the market by holding  $\alpha_{t+1}$  shares of the asset  $S$  in the interval  $(t, t+1]$ ,  $t \in \mathbb{N}_0$ , financing his/her purchases by borrowing (or lending to) the risk-free bond  $S^0$ . Therefore, the wealth process  $\{X_t\}_{t \in \mathbb{N}_0}$  is given by

$$X_t = x + \sum_{k=0}^{t-1} \alpha_{k+1} (S_{k+1} - S_k), \quad t \in \mathbb{N},$$

with  $X_0 = x$ . It can be shown that, for each  $t \in \mathbb{N}$ , there exists a unique minimal martingale measure  $\mathbb{Q}^{(t)}$  on  $\mathcal{F}_t$  (see [23] for details).

Define the  $\mathcal{F}_t$ -predictable ( $\mathcal{F}_{t-1}$ -adapted) process  $\{h_t\}_{t \in \mathbb{N}}$  given by

$$h_t = q_t \ln \left( \frac{q_t}{\mathbb{P}[A_t | \mathcal{F}_{t-1}]} \right) + (1 - q_t) \ln \left( \frac{1 - q_t}{1 - \mathbb{P}[A_t | \mathcal{F}_{t-1}]} \right), \quad t \in \mathbb{N}_0,$$

with

$$A_t = \{\omega : \xi_t(\omega) = \xi_t^u\} \quad \text{and} \quad q_t = \frac{1 - \xi_t^d}{\xi_t^u - \xi_t^d} = \mathbb{Q}^{(t)}[A_t | \mathcal{F}_{t-1}].$$

In [23] (see, also, [24]) it is shown that the random field  $U : \Omega \times \mathbb{N}_0 \times \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$U_t(x) = -\exp\left(-x + \sum_{k=1}^t h_k\right),$$

is a forward exponential performance. We, also, consider the inverse  $U^{-1}$  of  $U$  given by

$$U_t^{-1}(y) = -\ln(-y) - \sum_{k=1}^t h_k,$$

for  $y \in (-\infty, 0)$  and  $\{h_t\}_{t \in \mathbb{N}_0}$  as above.

For  $t \in \mathbb{N}_0$ , we define the (single-period) iterative forward price functional  $\mathcal{E}^{(t,t+1)} : \mathbb{L}^\infty(\mathcal{F}_{t+1}) \rightarrow \mathbb{L}^\infty(\mathcal{F}_t)$ , given by

$$\mathcal{E}^{(t,t+1)}(C) = \mathbb{E}_{\mathbb{Q}^{(t+1)}} \left[ -U_{t+1}^{-1} \left( \mathbb{E}_{\mathbb{Q}^{(t+1)}} [U_{t+1}(-C) | \mathcal{F}_t \vee \mathcal{F}_{t+1}^S] \right) \middle| \mathcal{F}_t \right],$$

for any  $C \in \mathbb{L}^\infty(\mathcal{F}_{t+1})$ . Similarly, for  $t < t'$  and  $C \in \mathbb{L}^\infty(\mathcal{F}_{t'})$  we define the (multi-step) forward pricing functional  $\mathcal{E}^{(t,t')} : \mathbb{L}^\infty(\mathcal{F}_{t'}) \rightarrow \mathbb{L}^\infty(\mathcal{F}_t)$  by

$$\mathcal{E}^{(t,t')}(C) = \mathcal{E}^{(t,t+1)} \left( \mathcal{E}^{(t+1,t+2)} \left( \dots \left( \mathcal{E}^{(t'-1,t')}(C) \right) \right) \right).$$

**Proposition 5.1.** *Let  $\rho(\cdot; t) : \mathbb{L}^\infty(\mathcal{F}_t) \rightarrow \mathbb{R}$  be defined by*

$$\rho(C; t) = \mathcal{E}^{(0,t)}(C).$$

*Then, the mapping  $\rho : \mathcal{L} = \cup_{t \in \mathbb{N}_0} \mathbb{L}^\infty(\mathcal{F}_t) \rightarrow \mathbb{R}$ , defined by*

$$\rho(C) = \rho(C; t_C)$$

*for  $t_C = \inf \{t \geq 0 : C \in \mathcal{F}_t\}$  is a maturity-independent convex risk measure.*

The statement of the Proposition follows from an argument analogous to the one in the proof of Proposition 4.10. For a detailed exposition of all steps, see [23].

## 6. SUMMARY AND FUTURE RESEARCH

The goals of the present paper are two-fold:

- (1) to bring forth and illustrate the concept of maturity-independent risk measures, and
- (2) to provide a class of such measures.

Two examples - one defined on a finite probability space and the other in an Itô-process setting - are given. Their analysis shows that, while plausible and simple from decision-theoretic point of view, the notion of maturity independence is non trivial and reveals an interesting structure .

One of the major sources of appeal of the theory of maturity-independent risk measures is, in our opinion, the fact that it opens a venue for a wide variety of research opportunities both from the mathematical, as well as the financial points of view. One of these directions, which we intend to pursue in forthcoming work (see [33]), follows the link between

maturity independence and forward performance processes in the direction opposite to the one explored here: while forward entropic risk measures provide a wide class of examples of maturity-independent risk measures, it is natural to ask whether there are any others. In other words, we would like to give a full characterization of maturity-independent risk measures arising from performance random fields. Such a characterization would not only complete the outlined theory from the mathematical point of view; it would also provide a firm decision-theoretic foundation for the sister theory of forward performance processes.

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